Knot Polynomials

Dr. Anton Kaul Kyler Andolina Vanessa Garcia Sanchez
Department of Mathematics, California Polytechnic State University, San Luis Obispo

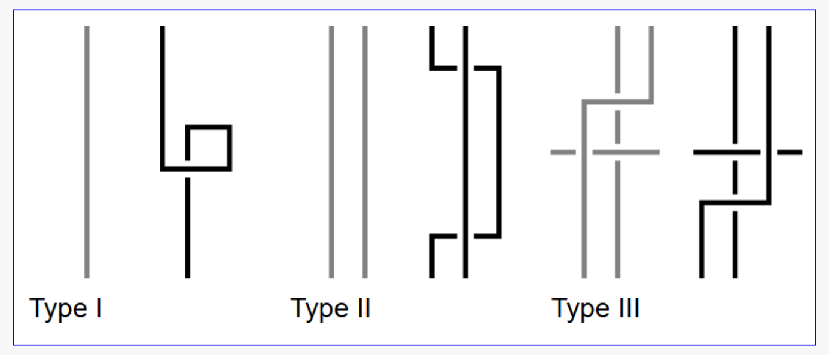
Introduction

Natural Question: If you are presented with two diagrams of a knot, is there a way to determine if the knots are equivalent?

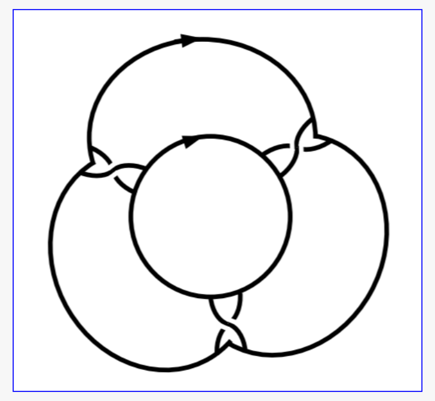
Approach: One way of tackling this question is by considering the first homology of the knot complement $S^3 \backslash K$. However, issues arise immediately since the homology group of any knot complement is $\mathbb{Z}$. Another way of approaching this question is by calculating the linking number of a knot. In our project, we explore and compare Alexander's method (as shown in "Topological Invariants of Knots and Links") with an algebraic topology construction. In the latter we will use the first homology of the infinite cyclic cover of the knot complement $H_1(\tilde{X})$.

Definitions

- A **knot** K is a simple closed curve in Euclidean 3-space.
- The **diagram** of a knot K is the projection of the knot onto the plane.
- We say two knots K_0 and K_1 are **ambiently isotopic** if
  1. There exists an orientation preserving homeomorphism f: $S^3 \to$ S^3 with $f(K_0) = K_1$
  2. such that f is homeotopic to Id_{S^3} via a homotopy such that f$_t$: $S^3 \to$ S^3 is an isomorphism for all $t \in [0, 1]$
- A **Reidemeister move** is a local move done on the diagram of the knot. The moves can be categorized into one of three types:



- Let p: $E \to$ S^3 be a map. We call p a **covering map** if there is an open cover $\{U_\alpha | \alpha \in I\}$ of X such that for each $\alpha \in I$, the preimage $p^{-1}(\{U_\alpha\})$ is the disjoint union of open sets, each mapped homeomorphically to U_α by p.
- We call $\pi_1(S^3 \backslash K)$ the **knot group**. This is read as "the fundamental group of the knot complement" and is defined as the group of the equivalence classes under homotopy of the loops contained in the knot complement.
- A **Seifert Surface** M for a knot is an orientable surface whose boundary is the knot itself.
  E.g. The picture below shows the seifert surface for the trefoil.



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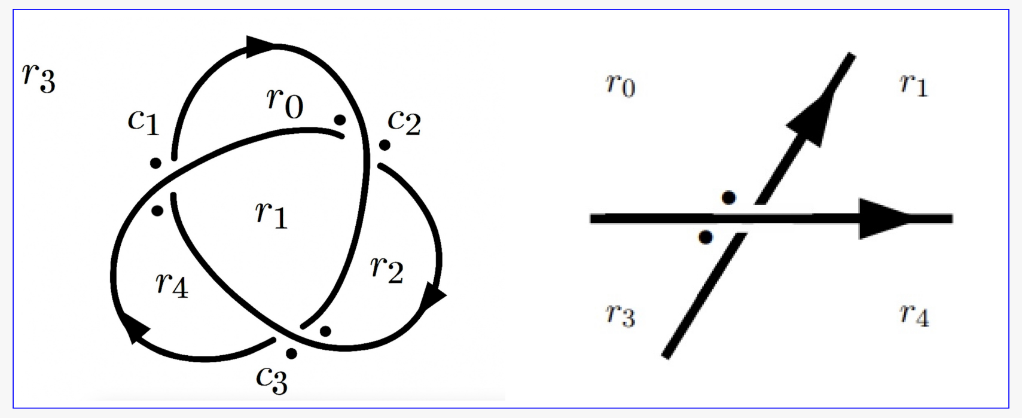
J. W. Alexander's Method

1 Let **D** be a proper diagram of a knot **K**, we will use the Trefoil Knot 3_1 as our example:



1. Let $C_1, C_2, ...C_n$ be each crossing of D and let $R_1, R_2, ...R_{n+2}$ be the regions of D, such that regions with indices one apart from each other are always separated by an edge of D and not just a corner. Each crossing will have the form as in the figure, and we will create equations $C_1(t), ...C_n(t)$, one for each crossing as depicted:

$$c_1(t) = tr_0 - tr_3 + r_4 - r_1 = 0$$
$$c_2(t) = tr_0 - tr_1 + r_4 - r_2 = 0$$
$$c_3(t) = tr_0 - tr_2 + r_4 - r_3 = 0$$

2. Then let **M** be the matrix obtained from our system of equations, each row corresponds to a crossing and each column corresponds to a region. We can then remove two adjacent columns R_i, R_j (corresponding to two regions separated by an edge) to obtain the square matrix $M_{i,j}$:

$$\begin{bmatrix} t & -1 & 0 & -t & 1 \\ t & -t & -1 & 0 & 1 \\ t & 0 & -t & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} t & -1 & 0 \\ t & -t & -1 \\ t & 0 & -t \end{bmatrix}$$

3. Then take the determinant of $M_{i,j}$ to obtain a polynomial and scale the polynomial accordingly until there is a positive constant term. The resulting polynomial is our invariant $\Delta(t)$:

$$\text{Det}(\begin{bmatrix} t & -1 & 0 \\ t & -t & -1 \\ t & 0 & -t \end{bmatrix}) = \text{t} \begin{bmatrix} -t & -1 \\ 0 & -t \end{bmatrix} + \begin{bmatrix} t & -1 \\ t & -1 \end{bmatrix} = t^3 - t^2 + t = t(t^2 - t + 1)$$

$$\Delta(t) = t^2 - t + 1$$

References

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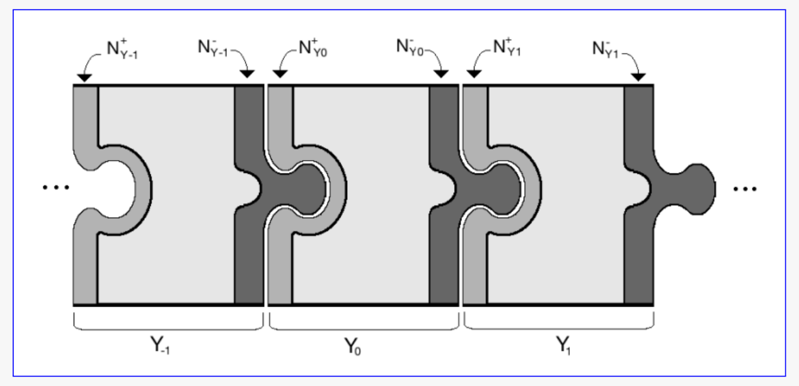
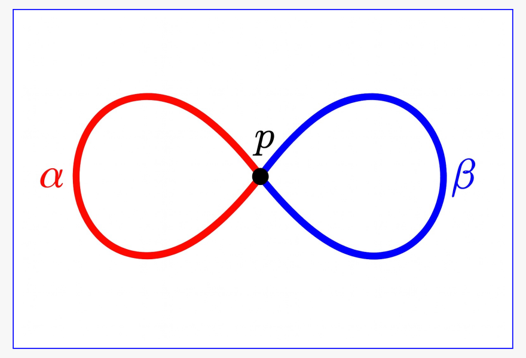
Infinite Cyclic Cover of $\tilde{X}$

Before we delve into the next approach, we must construct the infinite cyclic cover of th knot complement. First, we build the seifert surface of a knot and thicken the surface with a bicollar. Next, we make countably many copies of the seifret surface complement. Since these 2 surfaces overlap, we can glue them together along the corresponding sides. Below we include a snippet of the construction –for explicit details please refer to our last reference.

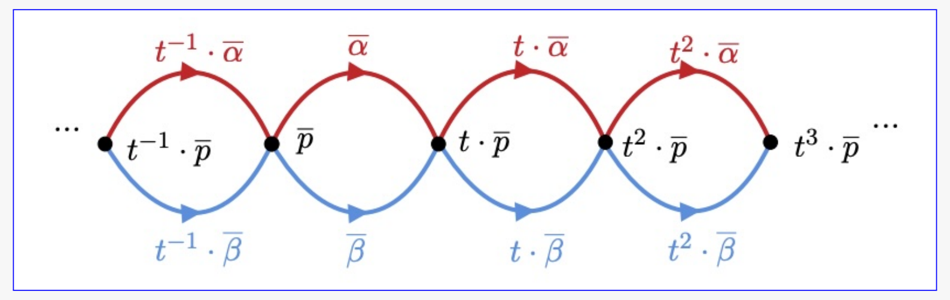


An Algebraic Topology Approach to $\tilde{X}$

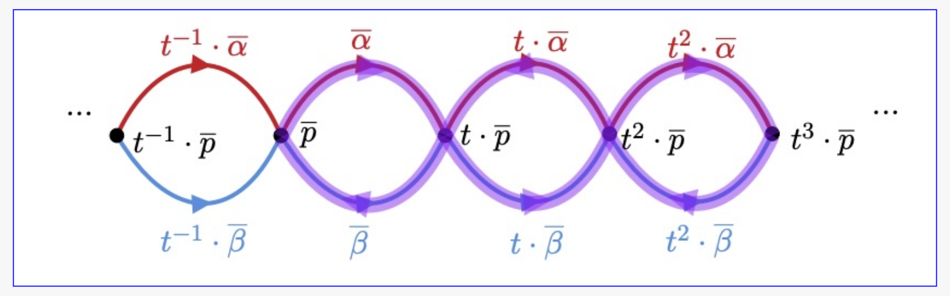
- First Let **X** be the knot complement of the trefoil in S^3 we have shown π_1(X) has 2 generators, call them α and β. X takes the form of a handlebody with these two generators, and we can create a deformation retract to obtain the skeleton:



- This deformation retract is a wedge of two circles, or two 1-cells, denoted α and β, attached to a singular 0-cell, denoted p. We then lift p up into $\tilde{X}$ to obtain infinite copies $...t^{-2}\bar{p}, t^{-1}\bar{p}, \bar{p}, t^1\bar{p}, t^2\bar{p}...$ and connect those infinite copies of p with the lifts of α and β to obtain the infinite skeleton:



- we now have an infinite 1-skeleton, the first homology group of which is merely the module of Laurent Polynomials with integer coefficients. Now starting at $\bar{p}$ we will glue a 2-cell along $\tilde{X}$ using our single relator: $\alpha\beta\alpha\beta^1\alpha^1\beta^1$ from $\pi_1(X)$, effectively killing that loop in the homology group, making it the only relator. The resulting 2-skeleton is our infinite cyclic cover $\tilde{X}$:



- now we can clearly see with this example with the trefoil we have:

$$H_1(\tilde{X}) \simeq \frac{\Lambda}{(t^2 - t + 1)}$$

where Λ is $\mathbb{Z}[x, x^1]$ the module of Laurent Polynomials, i.e those with positive and negative exponents and integer coefficients.