Counting Distinct Images of Matrices under Weak Compositions

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Introduction

We begin with a simple combinatorial question: Given a matrix M, how many distinct vectors do we get by adding up n of the columns? If we call this value $f(n)$, we define a sequence of numbers. We will use results from linear algebra and abstract algebra to prove that these sequences are eventually polynomial, and find a formula for the sequences using generating functions. This question was motivated by a problem in representation theory, which we present as an application.

Definitions and Preliminary Results

Definition. A *weak composition* of n into k parts where n is a non-negative integer and k is a positive integer is a k-tuple of non-negative integers that add up to n.

Definition. The set $\mathbb{Q}[x_1, x_2, \ldots, x_k]$ is the set of all polynomials in $x_1, \ldots, x_k$ with rational coefficients.

Definition. Let $I = \langle p_1, \ldots, p_t \rangle$ denote the set of all elements $\sum_{i=1}^{t} q_i p_i$ where $q_i \in \mathbb{Q}[x_1, \ldots, x_k]$. If every p_i is a monomial, I is a *monomial ideal*.

Result. Every ideal $I \subseteq \mathbb{Q}[x_1, \ldots, x_k]$ has a Gröbner basis, $G = \{g_1, \ldots, g_t\}$. Every polynomial $f \in \mathbb{Q}[x_1, \ldots, x_k]$ has a unique remainder when divided by G.

Opening Example

Consider the following matrix

$$\begin{bmatrix} 3 & 2 & 3 & 2 & 4 \\ 3 & 4 & 4 & 0 & 4 \end{bmatrix}.$$

The image given by the composition $(0, 1, 0, 0, 1)$ is

$$0\begin{bmatrix}3\\3\end{bmatrix} + 1\begin{bmatrix}2\\4\end{bmatrix} + 0\begin{bmatrix}3\\4\end{bmatrix} + 0\begin{bmatrix}2\\0\end{bmatrix} + 1\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix}.$$

Similarly, the image given by $(0, 0, 2, 0, 0)$ is

$$0\begin{bmatrix}3\\3\end{bmatrix} + 0\begin{bmatrix}2\\4\end{bmatrix} + 2\begin{bmatrix}3\\4\end{bmatrix} + 0\begin{bmatrix}2\\0\end{bmatrix} + 0\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix}.$$

Note these two compositions give the same image.

Translating to Polynomials

There is a natural correspondence between weak compositions of n into k parts and monomials of degree n in $\mathbb{Q}[x_1, \ldots, x_k]$.

$$(c_1, c_2, c_3, c_4, c_5) \quad \longleftrightarrow \quad x_1^{c_1} x_2^{c_2} x_3^{c_3} x_4^{c_4} x_5^{c_5}$$

Define M' to be M appended with a row of 1's:

$$M' = \begin{bmatrix} & M & \\ 1 & \cdots & 1 \end{bmatrix}.$$

Any two weak compositions that map to the same image have that their difference corresponds to a vector in the nullspace of M'. For example,

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \end{bmatrix}^{t} \in \operatorname{null} \begin{bmatrix} 3 & 2 & 3 & 2 & 4 \\ 3 & 4 & 4 & 0 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We can encode these relations into polynomials as follows

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \end{bmatrix}^{t} \quad \longleftrightarrow \quad x_2^1 x_5^1 - x_3^2.$$

We use the nullspace to encode all of these relations and form the polynomial ideal

$$I = \langle \{\mathbf{x}^{\alpha} - \mathbf{x}^{\beta} : \alpha - \beta \in \operatorname{null} M'\} \rangle.$$

Now, we can count how many cosets contain a monomial of degree n in the quotient

$$\mathbb{Q}[x_1, \ldots, x_k]/I$$

Gröbner Bases and Monomial Ideals

The ideal I has a Gröbner basis

$$I = \langle g_1, \ldots, g_t \rangle.$$

By properties of Gröbner bases, monomials are in the same coset if and only if they have the same remainder when divided by $\{g_1, \ldots, g_t\}$.

We form the monomial ideal

$$J = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t) \rangle,$$

quotient once again, and count cosets in

$$\mathbb{Q}[x_1, \ldots, x_k]/J$$

containing a monomial of degree n.

Stanley Decompositions

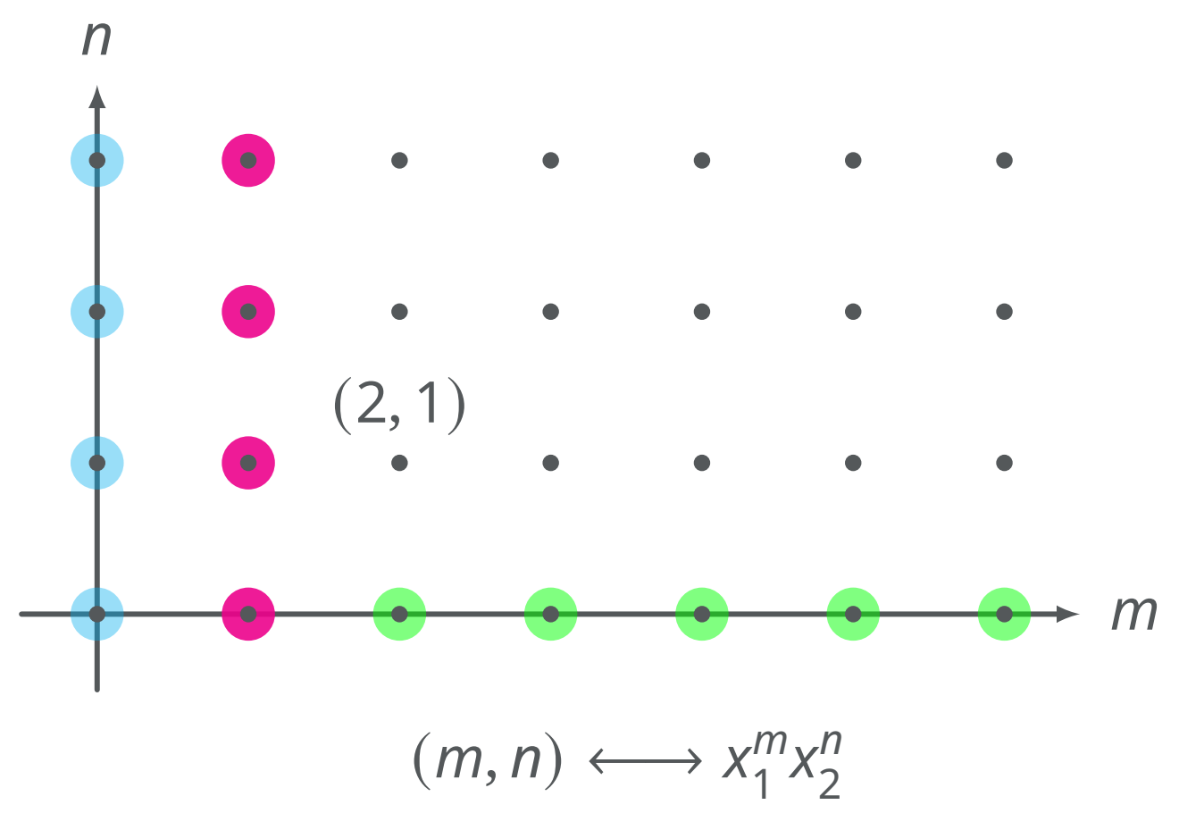
For a weak composition α, denote $\mathbf{x}^{\alpha} = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$. A *Stanley decomposition* is a way of expressing a polynomial ring as a direct sum of vector spaces,

$$R \cong \bigoplus_{\alpha \in F} \mathbf{x}^{\alpha} \mathbb{Q}[\mathbf{X}_{\alpha}],$$

where F is a set of weak compositions and $\mathbf{X}_{\alpha}$ is a subset of $\{x_1, \ldots, x_k\}$.

Here is an example of a Stanley decomposition:

$$\frac{\mathbb{Q}[x_1, x_2]}{\langle x_1^2 x_2 \rangle} \cong \mathbb{Q}[x_2] \oplus x_1 \mathbb{Q}[x_2] \oplus x_1^2 \mathbb{Q}[x_1].$$



We use a recursive algorithm from Sturmfels et al. to find the decomposition of $\mathbb{Q}[x_1, \ldots, x_k]/J$. The number of monomials of degree n in one component is given by the following generating function.

$$\mathbf{x}^{\alpha} k[\mathbf{X}_{\alpha}] \quad \longleftrightarrow \quad \frac{t^{|\alpha|}}{(1-t)^{|\mathbf{X}_{\alpha}|}}.$$

Results

For any matrix M, the sequence $f(n)$ is given by the generating function

$$\sum_{n=0}^{\infty} f(n) x^n = \sum_{\alpha \in F} \frac{t^{|\alpha|}}{(1-t)^{|\mathbf{X}_{\alpha}|}}$$

where F is given by a Stanley decomposition. Thus,

$$f(n) = \sum_{\alpha \in F} \binom{n - |\alpha| + |\mathbf{X}_{\alpha}| - 1}{|\mathbf{X}_{\alpha}| - 1},$$

and it follows that $f(n)$ satisfies a polynomial for

$$n \geq \max(\{|\alpha| - |\mathbf{X}_{\alpha}| + 1| : \alpha \in F\}).$$

Application to Representation Theory

How many representation of S_3 arising from group actions have exactly n orbits? First, we find the breakdown of the action of S_3 on cosets of its subgroups.

	$\langle e \rangle$	$\langle (12) \rangle$	$\langle (123) \rangle$	S_3
$\chi^{(1,1,1)}$	1	1	1	1
$\chi^{(2,1)}$	2	1	0	0
$\chi^{(3)}$	1	0	1	0

Table: Breakdowns of Coset Representations of S_3

Now, we find $f(n)$ for the table. Using our algorithm, we find $I = \langle x_1 x_4^2 - x_2^2 x_3 \rangle$, and $J = \langle x_1 x_4^2 \rangle$. The Stanley decomposition is given by

$$\mathbb{Q}[x_1, x_2, x_3, x_4]/J \cong$$

$$\mathbb{Q}[x_1, x_2, x_3] \oplus x_4 \mathbb{Q}[x_1, x_2, x_3] \oplus x_4^2 \mathbb{Q}[x_2, x_3, x_4].$$

This gives us the desired formula:

$$f(n) = \binom{n+2}{2} + \binom{n+1}{2} + \binom{n}{2}$$

$$= (1/2)(3n^2 + 3n + 2)$$

Acknowledgments and References

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